HEDEN'S BOUND ON THE TAIL OF A VECTOR SPACE PARTITION

SASCHA KURZ*

ABSTRACT. A vector space partition of $\mathbb{F}_q^v$ is a collection of subspaces such that every non-zero vector is contained in a unique element. We improve a lower bound of Heden, in a subcase, on the number of elements of the smallest occurring dimension in a vector space partition. To this end, we introduce the notion of q^r-divisible sets of k-subspaces in $\mathbb{F}_q^v$. By geometric arguments we obtain non-existence results for these objects, which then imply the improved result of Heden.

1. INTRODUCTION

Let $q > 1$ be a prime power, $\mathbb{F}_q$ be the finite filed with q elements, and v a positive integer. A *vector space partition* $\mathcal{P}$ of $\mathbb{F}_q^v$ is a collection of subspaces with the property that every non-zero vector is contained in a unique member of $\mathcal{P}$. If $\mathcal{P}$ contains m_d subspaces of dimension d, then $\mathcal{P}$ is of type $k^{m_k} \dots 1^{m_1}$. We may leave out some of the cases with $m_d = 0$. Subspaces of dimension d are also called *d-subspaces*. 1-subspaces are called *points*, $(v-1)$-subspaces are called *hyperplanes*, and each k-subspace contains $\begin{bmatrix}k\\1\end{bmatrix}_q := \frac{q^k-1}{q-1}$ points. So, in a vector space partition $\mathcal{P}$ each point of the ambient space $\mathbb{F}_q^v$ is covered by exactly one point of one of the elements of $\mathcal{P}$. An example of a vector space partition is given by a *k-spread* in $\mathbb{F}_q^v$, where $\begin{bmatrix}v\\1\end{bmatrix}_q/\begin{bmatrix}k\\1\end{bmatrix}_q$ k-subspaces partition the set of points of $\mathbb{F}_q^v$. The corresponding type is given by k^{m_k}, where $m_k = \begin{bmatrix}v\\1\end{bmatrix}_q/\begin{bmatrix}k\\1\end{bmatrix}_q$. If k divides v then considering the points of $\mathbb{F}_{q^k}^{v/k}$ as k-dimensional subspaces over $\mathbb{F}_q$ gives a construction of k-spreads. If k does not divide v, then no k-spreads exist. Vector space partitions of type $k^{m_k}1^{m_1}$ are known under the name *partial k-spreads*. More precisely, a partial k-spread in $\mathbb{F}_q^v$ is a set $\mathcal{K}$ of k-subspaces such that each point of the ambient space $\mathbb{F}_q^v$ is covered at most by one of its elements. Adding the set of uncovered points, which are also called *holes*, gives a vector space partition of type $k^{m_k}1^{m_1}$. Maximizing $m_k = \#\mathcal{K}$ is equivalent to the minimization of m_1. If d_1 is the smallest dimension with $m_{d_1} \neq 0$, we call m_{d_1} the *length of the tail* and call the set of the corresponding d_1-subspace the *tail*. Vector space partitions with a tail of small length are of special interest. In [4] Olof Heden obtained:

Theorem 1. *(Theorem 1 in* [4]*) Let $\mathcal{P}$ be a vector space partition of type ${d_l}^{u_l} \dots {d_2}^{u_2}{d_1}^{u_1}$ of $\mathbb{F}_q^v$, where $u_1, u_2 > 0$ and $d_l > \dots > d_2 > d_1 \geq 1$.*

- *(i) If $q^{d_2-d_1}$ does not divide u_1 and if $d_2 < 2d_1$, then $u_1 \geq q^{d_1} + 1$;*
- *(ii) if $q^{d_2-d_1}$ does not divide u_1 and if $d_2 \geq 2d_1$, then either d_1 divides d_2 and $u_1 = \begin{bmatrix}d_2\\1\end{bmatrix}_q/\begin{bmatrix}d_1\\1\end{bmatrix}_q$ or $u_1 > 2q^{d_2-d_1}$;*
- *(iii) if $q^{d_2-d_1}$ divides u_1 and $d_2 < 2d_1$, then $u_1 \geq q^{d_2} - q^{d_1} + q^{d_2-d_1}$;*
- *(iv) if $q^{d_2-d_1}$ divides u_1 and $d_2 \geq 2d_1$, then $u_1 \geq q^{d_2}$.*

Moreover, in Theorem 2 and Theorem 3 he classified the possible sets of d_1-subspaces for $u_1 = q^{d_1} + 1$ and $u_1 = \begin{bmatrix}d_2\\1\end{bmatrix}_q/\begin{bmatrix}d_1\\1\end{bmatrix}_q$, respectively. The results were obtained using the theory of mixed perfect 1-codes, see e.g. [6].

In [2] the authors improved a lower bound of Heden on the size of inclusion-maximal partial 2-spreads by translating the underlying techniques into geometry. Here we improve Theorem 1(ii). The underlying geometric structure is the set $\mathcal{N}$ of d_1-subspaces of a vector space partition $\mathcal{P}$ of type ${d_l}^{u_l} \dots {d_2}^{u_2}{d_1}^{u_1}$. For d_1 this is just a set of points in $\mathbb{F}_q^v$. It can be shown that the existence of $\mathcal{P}$ implies $\#\mathcal{N} \equiv \#(\mathcal{N} \cap H) \pmod{q^{d_2-1}}$ for every hyperplane H of $\mathbb{F}_q^v$, see e.g. [7]. Taking a vector representation of the elements of $\mathcal{N}$ as columns of a generator matrix, we obtain a corresponding (projective) linear code $\mathcal{C}$ over $\mathbb{F}_q$. The modulo constraints for $\mathcal{N}$ are equivalent to the property that the Hamming weights of the codewords of $\mathcal{C}$ are divisible by q^{d_2-1}. The study of so-called divisible codes, where the Hamming weights of the codewords of a linear code are divisible by some factor $\Delta > 1$, was initiated by Harold Ward, see e.g. [9]. The MacWilliams identities, linking the weight distribution of a linear code with the weight distribution of its dual code, can be relaxed to a linear program. Incorporating some information about the weight distribution of a linear code may result in an infeasible linear program, which then certifies the non-existence of such a code. This technique is known under the name linear programming method for codes and was more generally developed for association schemes by Philip Delsarte [3]. In [8] analytic solutions of linear programs for projective q^r-divisible linear codes have been applied in order to compute upper bounds for partial k-spreads. Indeed, all currently known upper bounds for partial k-spreads can be deduced from this method, see [7] for a survey.

* Grant KU 2430/3-1 – *Integer Linear Programming Models for Subspace Codes and Finite Geometry* – German Research Foundation.

Here, we generalize the approach to the case $d_1 > 1$ by studying the properties of the set $\mathcal{N}$ of d_1-subspaces of a vector space partition $\mathcal{P}$ of $\mathbb{F}_q^v$ of type $d_l^{u_l}\dots d_2^{u_2}d_1^{u_1}$ in Section 2. It turns out that we have $\#\mathcal{N} \equiv \#(\mathcal{N}\cap H) \pmod{q^{d_2-d_1}}$ for every hyperplane H of $\mathbb{F}_q^v$, see Lemma 3, which we introduce as a definition of a $q^{d_2-d_1}$*-divisible* set of k-subspaces with trivial intersection. By elementary counting techniques we obtain a partial substitute for the MacWilliams identities, see the equations (1) and (2). These imply some analytical criteria for the non-existence of such sets $\mathcal{N}$, which are used in Section 3 to reprove Theorem 1. By an improved analysis we tighten Theorem 1 to Theorem 12. More precisely, the second lower bound of Theorem 1(ii) is improved. We close with some numerical results on the spectrum of the possible cardinalities of $\mathcal{N}$ and pose some open problems.

2. Sets of disjoint k-subspaces and their incidences with hyperplanes

For a positive integer k let $\mathcal{N}$ be a set of pairwise disjoint, i.e., having trivial intersection, k-subspaces in $\mathbb{F}_q^v$, where we assume that the k-subspaces from $\mathcal{N}$ span $\mathbb{F}_q^v$, i.e., v is minimally chosen. By a_i we denote the number of hyperplanes H of $\mathbb{F}_q^v$ with $\#(\mathcal{N}\cap H) := \#\{U\in\mathcal{N} : U \le H\} = i$ and set $n := \#\mathcal{N}$. Due to our assumption on the minimality of the dimension v not all n elements from $\mathcal{N}$ can be contained in a hyperplane. Double-counting the incidences of the tuples (H), (B_1,H), and (B_1,B_2,H), where H is a hyperplane and $B_1 \neq B_2$ are elements of $\mathcal{N}$ contained in H gives:

$$\sum_{i=0}^{n-1} a_i = \begin{bmatrix} v \\ 1 \end{bmatrix}_q, \quad \sum_{i=0}^{n-1} i a_i = n\cdot \begin{bmatrix} v-k \\ 1 \end{bmatrix}_q, \quad \text{and} \quad \sum_{i=0}^{n-1} i(i-1)a_i = n(n-1)\cdot \begin{bmatrix} v-2k \\ 1 \end{bmatrix}_q. \tag{1}$$

For three different elements B_1,B_2,B_3 of $\mathcal{N}$ their span $\langle B_1,B_2,B_3\rangle$ has a dimension i between $2k$ and $3k$. Denoting the number of corresponding triples by b_i, double-counting tuples (B_1,B_2,B_3,H), where H is a hyperplane and B_1,B_2,B_3 are pairwise different elements of $\mathcal{N}$ contained in H, gives:

$$\sum_{i=0}^{n-1} i(i-1)(i-2)a_i = \sum_{i=2k}^{3k} b_i \begin{bmatrix} v-i \\ 1 \end{bmatrix}_q \quad \text{and} \quad \sum_{i=2k}^{3k} b_i = n(n-1)(n-2). \tag{2}$$

Given parameters q, k, n, and v the so-called *(integer) linear programming method* asks for a solution of the equation system given by (1) and (2) with $a_i, b_i \in \mathbb{R}_{\ge 0}$ ($a_i, b_i \in \mathbb{N}$). If no solution exists, then no corresponding set $\mathcal{N}$ can exist. For $k=1$ the equations from (1) and (2) correspond to the first four MacWilliams identities, see e.g. [7].

If there is a single non-zero value a_i the system can be solved analytically.

Lemma 2. *If $a_i = 0$ for all $i \neq r > 0$ and $k < v$ in the above setting, then there exists an integer $s \ge 2$ with $v = sk$ and $\mathcal{N}$ is a k-spread. Additionally we have $r = \frac{q^{v-k}-1}{q^k-1}$.*

Proof. Solving (1) for r, a_r, and n gives $n = \frac{q^{2v-k}-q^v-q^{v-k}+1}{q^v-q^{v-k}-q^k+1}$. Writing $v = sk+t$ with $s,t\in\mathbb{N}$ and $0 \le t < k$ we obtain $n = \sum_{i=1}^{s} q^{v-ik} + \frac{q^{v-k+t}-q^{v-k}-q^t+1}{q^v-q^{v-k}-q^k+1}$. Since $n\in\mathbb{N}$ and $0 \le q^{v-k+t}-q^{v-k}-q^t+1 < q^v-q^{v-k}-q^k+1$ we have $q^{v-k+t}-q^{v-k}-q^t+1 = 0$ so that $t=0$ and $n = \frac{q^v-1}{q^k-1}$. Counting points gives that $\mathcal{N}$ partitions $\mathbb{F}_q^v$. □

We remark that $r=0$ forces $n\in\{0,1\}$ so that $\mathcal{N}$ is empty or consists of a single k-subspace in $\mathbb{F}_q^k$ and $v=k$ implies the latter case. So, these degenerated cases correspond to $s\in\{0,1\}$ in Lemma 2. As pointed out after [4, Theorem 2], such results can be proved in different ways. While the case that only one a_i is non-zero is rather special, we can show that many a_i are equal to zero in our setting.

Lemma 3. *Let $\mathcal{P}$ be a vector space partition of type $d_l^{u_l}\dots d_2^{u_2}d_1^{u_1}$ of $\mathbb{F}_q^v$, where $u_1,u_2>0$, and let $\mathcal{N}$ be the set of d_1-subspaces. Then, we have $\#\mathcal{N} \equiv \#(\mathcal{N}\cap H) \pmod{q^{d_2-d_1}}$ for every hyperplane H of $\mathbb{F}_q^v$.*

Proof. For each $U\in\mathcal{P}$ we have $\dim(U\cap H)\in\{\dim(U),\dim(U)-1\}$. So counting points in $\mathbb{F}_q^v$ and H gives the existence of integers a,a' with $m\cdot\begin{bmatrix} d_2 \\ 1 \end{bmatrix}_q + aq^{d_2} + u_1\begin{bmatrix} d_1 \\ 1 \end{bmatrix}_q = \begin{bmatrix} v \\ 1 \end{bmatrix}_q$ and $m\cdot\begin{bmatrix} d_2-1 \\ 1 \end{bmatrix}_q + a'q^{d_2-1} + u_1'q^{d_1-1} + u_1\begin{bmatrix} d_1-1 \\ 1 \end{bmatrix}_q = \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q$, where $m := \sum_{i=2}^{l} u_i$ and $u_1' := \#(\mathcal{N}\cap H)$. By subtraction we obtain $mq^{d_2-1} + aq^{d_2} - a'q^{d_2-1} + u_1q^{d_1-1} - u_1'q^{d_1-1} = q^{v-1}$, so that $u_1q^{d_1-1} \equiv u_1'q^{d_1-1} \pmod{q^{d_2-1}}$. □

Definition 4. Let $\mathcal{N}$ be a set of k-subspaces in $\mathbb{F}_q^v$. If there exists a positive integer r such that a_i is non-zero only if $\#\mathcal{N}-i$ is divisible by q^r and the k-subspaces are pairwise disjoint, then we call $\mathcal{N}$ q^r*-divisible*.

Using the notation of Lemma 3, $\mathcal{N}$ is $q^{d_2-d_1}$-divisible. As mentioned in the introduction, for $d_1 = 1$, taking the elements of $\mathcal{N}$ as columns of a generator matrix, we obtain a projective linear code, whose Hamming weights are divisible by q^{d_2-1}.

Example 5. For integers $k\ge 2$ and $r = ak+b$ with $0\le b<k$ let $\mathcal{N}$ be a k-spread of $\mathbb{F}_q^{(a+2)k}$. Starting from a $(a+2)k$-spread in $\mathbb{F}_q^{2(a+2)k}$ we obtain a vector space partition $\mathcal{P}$ by replacing one $(a+2)k$-dimensional spread

element with $\mathcal{N}$. From Lemma 3 and $q^r|q^{(a+2)k-k}=q^{(a+1)k}$ we deduce that the set $\mathcal{N}$ of k-subspaces is q^r-divisible. Its cardinality is given by $\begin{bmatrix}(a+2)k\\1\end{bmatrix}_q/\begin{bmatrix}k\\1\end{bmatrix}_q$.

Example 6. For integers $k\ge 2$ and $r\ge 1$ let $n=k+r$ and consider a matrix representation $M\colon \mathbb{F}_{q^n}\to\mathbb{F}_q^{n\times n}$ of $\mathbb{F}_{q^n}/\mathbb{F}_q$, obtained by expressing the multiplication maps $\mu_\alpha\colon\mathbb{F}_{q^n}\to\mathbb{F}_{q^n}$, $x\mapsto\alpha x$, which are linear over $\mathbb{F}_q$, in terms of a fixed basis of $\mathbb{F}_{q^n}/\mathbb{F}_q$. Then, all matrices in $M(\mathbb{F}_{q^n})$ are invertible and have mutual rank distance $d_{\mathrm{R}}(A,B):=\mathrm{rk}(A-B)=n$, see e.g. [7] for proofs of these and the subsequent facts. In other words, the matrices of $M(\mathbb{F}_{q^n})$ form a maximum rank distance code with minimum rank distance n and cardinality q^n.

Now let $\mathcal{B}\subseteq\mathbb{F}_q^{k\times n}$ be the matrix code obtained from $M(\mathbb{F}_{q^n})$ by deleting the last $n-k$ rows, say, of every matrix. Then $\mathcal{B}$ has cardinality minimum rank distance k. Hence, by applying the lifting construction $B\mapsto(I_k|B)$, where I_k is the $k\times k$ identity matrix, to $\mathcal{B}$ we obtain a partial k-spread $\mathcal{N}$ in $\mathbb{F}_q^v$ of size $q^n=q^{k+r}$. Since precisely the points outside the $(k+r)$-subspace $S=\left\{x\in\mathbb{F}_q^v : x_1=x_2=\dots=x_k=0\right\}$ are covered, $\mathcal{P}=\mathcal{N}\cup\{S\}$ is a vector space partition of $\mathbb{F}_q^{2k+r}$ and $\mathcal{N}$ is q^{k+r}-divisible with cardinality q^{k+r}.

From the first two equations of (1) we deduce:

Lemma 7. *For a q^r-divisible set $\mathcal{N}$ of k-subspaces in $\mathbb{F}_q^v$, there exists a hyperplane H with $\#(\mathcal{N}\cap H)\le n/q^k$.*

PROOF. Let i be the smallest index with $a_i\neq 0$. Then, the first two equations of (1) are equivalent to $\sum_{j\ge0}a_{i+q^rj}=\begin{bmatrix}v\\1\end{bmatrix}_q$ and $\sum_{j\ge0}(i+q^rj)\cdot a_{i+q^rj}=n\begin{bmatrix}v-k\\1\end{bmatrix}_q$. Subtracting i times the first equation from the second equation gives $\sum_{j>0}q^rja_{i+q^rj}=n\cdot\frac{q^{v-k}-1}{q-1}-i\cdot\frac{q^v-1}{q-1}$. Since the left-hand side is non-negative, we have $i\le\frac{q^{v-k}-1}{q^v-1}\cdot n\le\frac{n}{q^k}$. □

Stated less technical, the proof of Lemma 7 is given by the fact that the hyperplane with the minimum number of k-subspaces contains at most as many k-subspaces as the average number of k-subspaces per hyperplane.

Taking also the third equation of (1) into account implies a quadratic criterion:

Lemma 8. *Let $m\in\mathbb{Z}$ and $\mathcal{N}$ be a q^r-divisible set of k-subspaces in $\mathbb{F}_q^v$. Then, $\tau(n,q^r,q^k,m)\cdot q^{v-2k-2r}-m(m-1)\ge 0$, where $\tau(n,\Delta,u,m):=\Delta^2u^2m(m-1)-n(2m-1)u(u-1)\Delta+n(u-1)(n(u-1)+1)$.*

PROOF. With $y=q^{v-2k}$, $u=q^k$, and $\Delta=q^r$, we can rewrite the equations of (1) to $u^2y-1=(q-1)\sum_{i\in\mathbb{Z}}a_i$, $n\cdot(uy-1)=(q-1)\sum_{i\in\mathbb{Z}}ia_i$, and $n(n-1)\cdot(y-1)=\sum_{i\in\mathbb{Z}}i(i-1)a_i$. $(n-m\Delta)(n-(m-1)\Delta)$ times the first minus $2n-(2m-1)\Delta-1$ times the second plus the third equation gives $y\cdot\tau(n,\Delta,u,m)-\Delta^2m(m-1)=(q-1)\sum_{i\in\mathbb{Z}}(n-m\Delta-i)(n-(m-1)\Delta-i)a_i=(q-1)\sum_{h\in\mathbb{Z}}\Delta^2(m-h)(m-h+1)a_{n-h\Delta}\ge 0$. □

As a preparation we present another classification result:

Lemma 9. *If $\mathcal{N}$ is a q-divisible set of k-subspaces in $\mathbb{F}_q^v$ of cardinality q^k+1, then $\mathcal{N}$ partitions $\mathbb{F}_q^{2k}$.*

PROOF. Setting $c_i:=(q-1)a_{1+iq}$ and $l:=q^{k-1}-1$ we can rewrite the equations of (1) to $\sum_{i=0}^{l}c_i=q^v-1$, $\sum_{i=0}^{l}(1+iq)c_i=(q^k+1)\left(q^{v-k}-1\right)$, and $\sum_{i=0}^{l}iq(1+iq)c_i=(q^k+1)q^k\left(q^{v-2k}-1\right)$. Since $ql+1$ times the second minus $ql+1$ times the first minus the third equation gives $0\le\sum_{i=0}^{l}iq^2(l-i)c_i=-q^{k+1}\left(q^{v-2k}-1\right)$, we have $v=2k$. Every point of $\mathbb{F}_q^v$ is covered by an element from $\mathcal{N}$ due to $\begin{bmatrix}2k\\1\end{bmatrix}_q/\begin{bmatrix}k\\1\end{bmatrix}_q=q^k+1$. □

3. PROOF OF HEDEN'S RESULTS AND FURTHER IMPROVEMENTS

Let $\mathcal{P}$ be a vector space partition of type ${d_l}^{u_l}\dots{d_2}^{u_2}{d_1}^{u_1}$ of $\mathbb{F}_q^{v'}$, where $u_1,u_2>0$, $d_l>\dots>d_2>d_1\ge 1$. Let $\mathcal{N}$ be the set of d_1-subspaces and V be the subspace spanned by $\mathcal{N}$. By n we denote the cardinality of $\mathcal{N}$ and by a_i we denote the number of hyperplanes of V that contain exactly i elements from $\mathcal{N}$.

Assume that $q^{d_2-d_1}$ does not divide u_1. We have $\#(\mathcal{N}\cap H)\ge 1$ for every hyperplane H of V due to Lemma 3, so that Lemma 7 gives $u_1\ge q^{d_1}$. Thus, we have $u_1\ge q^{d_1}+1$. If $u=q^{d_1}+1$ then we can apply Lemma 9 for the classification of the possible sets $\mathcal{N}$. If $u_1<2q^{d_2-d_1}$ then for $a_i>0$ we have $i<q^{d_2-d_1}$ and $i\equiv u_1 \pmod{q^{d_2-d_1}}$ so that we can apply Lemma 2. Thus, either d_2 divides d_1 and $u_1=(q^{d_2}-1)/(q^{d_1}-1)$ or $u_1>2q^{d_2-d_1}$. The first case can be attained by a d_2-spread where one d_2-subspace is replaced by a d_1-spread, see Example 5. We remark that no assumption on the relation between d_2 and d_1 is used in our derivation. However, if $d_2<2d_1$ then d_1 cannot divide d_2 and $q_1^d+1>2q^{d_2-d_1}$.

Assume that $q^{d_2-d_1}$ divides u_1. Setting $\Delta=q^{d_2-d_1}$, $u=q^{d_1}$, $n=\Delta l$, and $m=l$† for some integer l, we conclude $\tau(n,\Delta,u,m)=\Delta l(\Delta l-\Delta u+u-1)\ge 0$ from Lemma 8, so that $l\ge\left\lceil u-\frac{u}{\Delta}+\frac{1}{\Delta}\right\rceil$. The right-hand side is equal to $u=q^{d_1}$ if $d_2\ge 2d_1$ and to $u-u/\Delta+1=q^{d_1}-q^{2d_1-d_2}+1$ otherwise, which is equivalent to $n\ge q^{d_2}$ and $n\ge q^{d_2}-q^{d_1}+q^{d_2-d_1}$. We remark that equality is achievable in the latter case via the 2-weight codes constructed in [1] (with parameters $n'=d_1$ and $m=d_2-d_1$). We do not know whether the corresponding $q^{d_2-d_1}$-divisible set of d_1-subspaces can be realized as a vector space partition of $\mathbb{F}_q^v$.‡ For the first case see Example 6.

†The choice for m can be obtained by minimizing $\tau(n,\Delta,u,m)$, i.e., solving $\frac{\partial\tau(n,\Delta,u,m)}{\partial m}=0$ and rounding.

‡A suitable test case might be to decide whether a vector space partition of type $4^43^{135}2^6$ exists in $\mathbb{F}_2^{10}$.

The above comprises [4, Theorems 1-4]. Given the stated examples, just Theorem 1(ii), for the case where d_1 does not divide d_2, leaves some space for improving the lower bound on u_1. To that end we analyze Lemma 8 in more detail. Since the statements look rather technical and complicated we first give a justification for the necessity of this fact. Via the quadratic inequality of Lemma 8 intervals of cardinalities can be excluded for different values of the parameter m. However, some cardinalities are indeed feasible. If $r = ak + b$ with $0 \le b < k$ then the two constructions from Example 5 and Example 6 give q^r-divisible set of k-subspaces of cardinality $\left[{(a+2)k \atop 1}\right]_q / \left[{k \atop 1}\right]_q$ and q^{k+r}, respectively. For $q = 2$, $r = 3$, $k = 2$ the cardinalities of these two examples are given by 21 and 32. In general, each two q^r-divisible sets $\mathcal{N}_1$ and $\mathcal{N}_2$ of k-subspaces can be combined to a q^r-divisible set of k-subspaces of cardinality $\#\mathcal{N}_1 + \#\mathcal{N}_2$. Since $\left[{(a+2)k \atop 1}\right]_q / \left[{k \atop 1}\right]_q$ and q^{k+r} are coprime there exists some integer $F_q(k, r)$ such that q^r-divisible sets of k-subspaces exist for every cardinality $n > F_q(k, r)$. Below that number some cardinalities can be excluded, but their density decreases with increasing n. Our numerical example is continued after the proof of Theorem 12.

Proposition 10. *Let $\mathcal{N}$ be a q^r-divisible set of k-subspaces in $\mathbb{F}_q^v$, $u = q^k$ and $\Delta = q^r$. Then, $n \notin \left[1, \frac{q^{k+r}-1}{q^r-1}\right)$ and*

$$n \notin \left[\left\lceil \frac{1}{u-1} \cdot \left(\Delta u m - \frac{\Delta u + 1}{2} - \frac{1}{2}\sqrt{\omega}\right)\right\rceil, \left\lfloor \frac{1}{u-1} \cdot \left(\Delta u m - \frac{\Delta u + 1}{2} + \frac{1}{2}\sqrt{\omega}\right)\right\rfloor\right],$$

where $\omega = (\Delta u - 2m)^2 + \left(2\Delta u + 1 - 4m^2\right)$, for all $m \in \mathbb{N}$ with $2 \le m \le \left\lfloor \frac{\Delta u}{4} + \frac{1}{2} + \frac{1}{4\Delta u}\right\rfloor$.

PROOF. We set $\overline{\Delta} = \Delta u$ and $\overline{n} = n(u-1)$ so that $\tau(n, \Delta, u, m) = \overline{\Delta}^2 m(m-1) - \overline{n}\overline{\Delta}(2m-1) + \overline{n}(\overline{n}+1)$. We have $\tau(n, \Delta, u, m) \le 0$ iff $\left|\overline{n} - \overline{\Delta}m + \frac{\overline{\Delta}+1}{2}\right| \le \frac{1}{2}\sqrt{\overline{\Delta}^2 - 4m\overline{\Delta} + 2\overline{\Delta} + 1}$ and $m \le \frac{\overline{\Delta}}{4} + \frac{1}{2} + \frac{1}{4\overline{\Delta}}$. Rewriting and applying Lemma 8 with $1 \le m \le \left\lfloor \frac{\Delta u}{4} + \frac{1}{2} + \frac{1}{4\Delta u}\right\rfloor$ gives the result since $m(m-1) > 0$ for $m \ge 2$. □

Proposition 11. *Let $\mathcal{N}$ be a q^r-divisible set of k-subspaces in $\mathbb{F}_q^v$, where $r = ak + b$ with $a, b \in \mathbb{N}$, $0 < b < k$ and $a \ge 1$. Then, $n \ge \frac{q^{(a+2)k}-1}{q^k-1} = q^r \cdot q^{k-b} + \frac{q^r \cdot q^{k-b}-1}{q^k-1} = \Delta q^{k-b} + q^k\Theta + 1$, where $\Delta := q^r$ and $\Theta := \frac{q^{ak}-1}{q^k-1}$.*

PROOF. From Lemma 2 we conclude $n \ge 2q^r$ and set $u = q^k$. For $2 \le m \le q^{k-b}$ we have $2\Delta u + 1 - 4m^2 > 0$, so that Proposition 10 gives $n \notin \left[\left\lceil \frac{\Delta u(m-1)-1/2+m}{u-1}\right\rceil, \left\lfloor \frac{\Delta u m-1/2-m}{u-1}\right\rfloor\right]$. Since $\Delta(m-1) \le \left\lceil \frac{\Delta u(m-1)-1/2+m}{u-1}\right\rceil = \Delta(m-1) + \left\lceil \frac{\Delta(m-1)-1/2+m}{u-1}\right\rceil \le \Delta m$ and $\left\lfloor \frac{\Delta u m-1/2-m}{u-1}\right\rfloor = \Delta m + mq^b\Theta + \left\lfloor \frac{mq^b-1/2-m}{q^k-1}\right\rfloor = \Delta m + mq^b\Theta$, we conclude $n \notin \left[\Delta m, \Delta m + mq^b\Theta\right]$ for $2 \le m \le q^{k-b}$.

It remains to show $n \notin \left[\Delta m, \Delta m + mq^b\Theta + 1, \Delta(m+1) - 1\right] =: I_m$ for all $2 \le m \le q^{k-b} - 1$. If $n \in I_m$, then we can write $n = \Delta m + mq^b\Theta + x$ with $x \ge 1$ and $mq^b\Theta + x < \Delta$, so that $q^k \cdot \left(mq^b\Theta + x\right) = \Delta m + mq^b\Theta + \left(xq^k - mq^b\right) < \Delta m + mq^b\Theta + x = n$, which contradicts Lemma 7. □

In other words, in the case of Theorem 1(ii), where $d_2 = ad_1 + b$ with $0 < b < d_1$ and $a, b \in \mathbb{N}$, we have $u_1 \ge q^{d_2-d_1} \cdot q^{d_1-b} + \frac{q^{(a+1)d_1}-1}{q^{d_1}-1} = \frac{q^{(a+2)d_1}-1}{q^{d_1}-1}$, which can be attained by an d_1-spread in $\mathbb{F}_q^{(a+2)d_1}$. Without the knowledge of b, we can state $u_1 \ge q \cdot q^{d_2-d_1} + \left\lceil \frac{q^{d_2+1}-1}{q^{d_1}-1}\right\rceil$, which also improves Theorem 1(ii) and is tight whenever $d_2 + 1$ is divisible by d_1. Summarizing our findings we obtain our main theorem:

Theorem 12. *For a non-empty q^r-divisible set $\mathcal{N}$ of k-subspaces in $\mathbb{F}_q^v$ the following bounds on $n = \#\mathcal{N}$ are tight.*

(i) We have $n \ge q^k + 1$ and if $r \ge k$ then either k divides r and $n \ge \frac{q^{k+r}-1}{q^k-1}$ or $n \ge \frac{q^{(a+2)k}-1}{q^k-1}$, where $r = ak + b$ with $0 < b < k$ and $a, b \in \mathbb{N}$.

(ii) Let q^r divide n. If $r < k$ then $n \ge q^{k+r} - q^k + q^r$ and $n \ge q^{k+r}$ otherwise.

For (i) the lower bounds are attained by k-spreads, see Example 5. For (ii) the second lower bound is attained by a construction based on lifted MRD codes, see Example 6. In the other case the 2-weight codes constructed in [1] attain the lower bound. Thus, Theorem 12 is tight and implies an improvement of Theorem 1(ii).

While the smallest cardinality of a non-empty q^r-divisible set of k-subspaces over $\mathbb{F}_q$ has been determined, the spectrum of possible cardinalities remains widely unknown. For $k = 1$ [7, Theorem 12] states that either $n > rq^{r+1}$ or there exist integers a, b with $n = a\left[{r+1 \atop 1}\right]_q + bq^{r+1}$ and bounds for the maximum excluded cardinality have been determined in [5]. However, Lemma 7 and Lemma 8, applied via Proposition 10, give restrictions going far beyond Theorem 12. For $q = 2$, $r = 3$, $k = 2$, and $n \le 81$ we exemplarily state that only $n \in \{21, 31, 32, 33, 42, 43, 44, 52, \dots, 55, 62, \dots, 66, 72, \dots, 78\}$ might be attainable. The mentioned constructions cover the cases $n \in \{21, 32, 42, 53, 63, 64, 74\} \subseteq \{21a + 32b : a, b \in \mathbb{N}\}$. Replacing the lines by their contained 3 points, we obtain 2^4-divisible sets of 1-subspaces in $\mathbb{F}_q^v$ of cardinality $3n$, for which two further exclusion criteria have been presented in [7], excluding the cases $n \in \{33, 44\}$. [7, Lemma 23] is based on a cubic polynomial obtained from (1) and (2), similar to the quadratic polynomial from Lemma 8 obtained from (1). Here, the presence of k additional b_i-variables may make the analysis more difficult for $k > 1$. For a q^r-divisible set $\mathcal{N}$ of 1-subspaces we have that $\mathcal{N} \cap H$ is q^{r-1}-divisible for every hyperplane H, which allows a recursive application of the linear programming method. For $k > 1$ we need to consider k-subspaces and $k-1$-subspaces in H, see [7, Section 6.3], which makes the bookkeeping more complicated.

The determination of the possible spectrum of cardinalities of q^r-divisible sets of k-subspaces remains an interesting open problem. Even for small parameters this might be challenging. A possible intermediate step is the determination of the number $F_q(k,r)$ being similar to the Frobenius number. Extending the small list of constructions is also worthwhile.

ACKNOWLEDGEMENT

I am very thankful for the comments of two anonymous reviewers, which helped to improve the paper.

REFERENCES

[1] J. Bierbrauer and Y. Edel, *A family of* 2*-weight codes related to BCH-codes*, Journal of Combinatorial Designs **5** (1997), no. 5, 391.
[2] A. Blokhuis, A.E. Brouwer, and H.A. Wilbrink, *Heden's bound on maximal partial spreads*, Discrete Mathematics **74** (1989), no. 3, 335–339.
[3] P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips research reports (1973), no. 10, 103.
[4] O. Heden, *On the length of the tail of a vector space partition*, Discrete Mathematics **309** (2009), no. 21, 6169–6180.
[5] D. Heinlein, T. Honold, M. Kiermaier, S. Kurz, and A. Wassermann, *Projective divisible binary codes*, The Tenth International Workshop on Coding and Cryptography 2017, 2017.
[6] M Herzog and J Schönheim, *Group partition, factorization and the vector covering problem*, Canad. Math. Bull **15** (1972), no. 2, 207–214.
[7] T. Honold, M. Kiermaier, and S. Kurz, *Partial spreads and vector space partitions*, Network Coding and Subspace Designs (M. Greferath, M.O. Pavčević, N. Silberstein, and A. Vazquez-Castro, eds.), Springer, 2018, pp. 131–170.
[8] S. Kurz, *Packing vector spaces into vector spaces*, The Australasian Journal of Combinatorics **68** (2017), no. 1, 122–130.
[9] H.N. Ward, *Divisible codes*, Archiv der Mathematik **36** (1981), no. 1, 485–494.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BAYREUTH, 95440 BAYREUTH, GERMANY
E-mail address: sascha.kurz@uni-bayreuth.de